Infinite Triangle and Hexagonal Lattice Networks of Identical Resistor

Septinus George Saa

National Public School 3 Jayapura, Penggalang Street No.59 Waena Jayapura-Papua INDONESIA, Phone/Fax: 62 967 571506

Abstract

The resistance between two arbitrary points in an infinite triangle and hexagonal lattice networks of identical resistor are calculated using Fourier series. For hexagonal networks, I discover a new method of calculating the resistance directly from the networks instead of using $\Delta - Y$ transformation, which is commonly used. The result are compared numerically to other authors that utilizing separation variable [1,2] or Green function [3].

1. Introduction

Problem of calculating resistance between two arbitrary lattices points in the infinite lattice of networks has been revive recently. Several methods have been introduced to solve the problem. We could find the old of this efforts and cited paper related with this topic in Zemanian paper [4]. One of the relatively new approaches is developed by Giulio Vanezian [1]. Vanezian used a superposition method and explicating the symmetry of the grid. The mathematical problem involves the solution of an infinite set of linear, inhomogeneous difference equations, which are solved by the method of separation variables. This method is developed further by D. Atkinson and F.J. van Steewijk [2] to calculate the resistance between two arbitrary nodes in infinite triangular and hexagonal lattices in two dimensions. In addition, for hexagonal lattices they used $\Delta - Y$ transformation of triangular lattices. Doyle and Snell presented the method to calculate the infinite electric networks using the random walks, though they did not show explicitly and numerically of the results [5]. Later on Jozsef Cserti used the lattice Green function to calculate the resistance of infinite networks of resistors [3].

In this paper I present the alternative approach, which was introduced by Krzysztof Giaro [6] who calculated the resistance between any two points in the infinite square lattice networks that utilizes the basic properties of Fourier series. The method has been extended for the infinite cubic lattice networks by Agus Wirawan [7]. I use this method to calculate the resistance between any two points in the infinite triangle and hexagonal lattices networks. For the hexagonal lattice, we use two methods: First, $\Delta - Y$ transformation and secondly the use of the Fourier series directly from the hexagonal networks. In the analysis, we use an orthogonal Cartesian coordinate system (one axis is horizontal and other is vertical) instead of a hexagonal or triangle coordinate system (one axis in horizontal and the other is inclined at 120^0) that often used in the triangle or hexagonal lattice analysis [2,3]. To some extent, the orthogonal Cartesian coordinate system is easier to follow both for triangular and hexagonal lattice compared to the other coordinate systems. The result are written in a double integral formulas equation (18), (36) and (43). These formulas look different from the formula derived by some authors [2,3]. However, I prove numerically that both results are actually same.

The paper is divided into two sections. In the first section, I will perform analysis of the triangle infinite networks of resistor to calculate the resistance between any two points in the networks, and in the second section, I will do the analysis for the hexagonal infinite networks of resistor.

2. Infinite triangle lattice networks of identical resistor

Consider an infinite triangle lattice formed from equal resistors. Let *(j,k)* be the node (point) that is *j* unit in the horizontal, and *k* is in the vertical direction from an arbitrarily chosen origin. The point has six nearest neighboring points, which is the small part of infinite triangle lattice (networks of resistors). The coordinate of these points are shown in Fig. 1. It is assumed that a current of *I* amperes enter the *(0,0)* node from a source outside the lattice and leaves at *(m,n)*.

$$I(j,k) = \begin{cases} I & \text{If } j = k = 0 \\ -I & \text{If } j = m \text{ and } k = n \\ 0 & \text{For all other case} \end{cases}$$

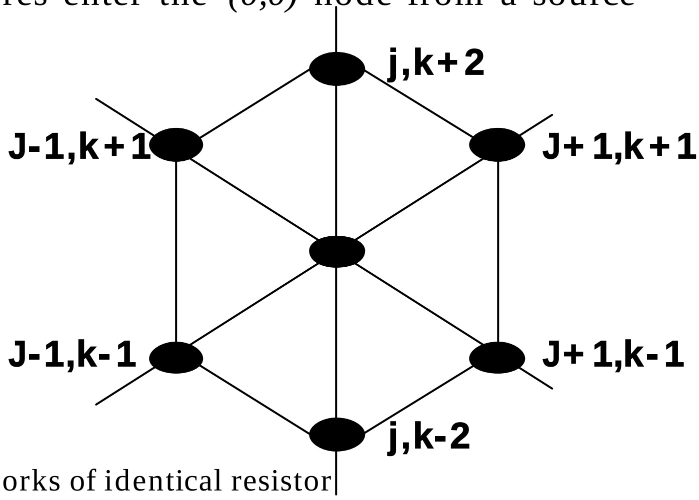

Figure.1. Infinite triangle lattice networks of identical resistor

Let the potential at the point *(j,k)* is denoted by $V_{j,k}$ volts, and $V_{j,k}$ tends to zero at branch point very far from *(0,0)* and *(m,n)*. By a combination of Kirchoff's first rule and Ohm's laws, we have,

$$rI_{j,k} = \left(V_{j,k} - V_{j+1,k+1}\right) + \left(V_{j,k} - V_{j+1,k-1}\right) + \left(V_{j,k} - V_{j-1,k-1}\right) + \left(V_{j,k} - V_{j-1,k+1}\right) + \left(V_{j,k} - V_{j,k+1}\right) + \left(V_{j,k} - V_{j,k-2}\right)$$

Or

$$rI_{j,k} = 6V_{j,k} - \left(V_{j+1,k+1} + V_{j+1,k-1} + V_{j-1,k-1} + V_{j-1,k+1} + V_{j,k+2} + V_{j,k-1}\right) \qquad [1]$$

Let $\Phi_{j,k}$ be any scalar function define on all vertices of our network. I will denote Λ a linear operator satisfying the following equation,

$$\Lambda\Phi_{j,k} = \frac{1}{6}\left(\Phi_{j+1,k+1} + \Phi_{j+1,k-1} + \Phi_{j-1,k-1} + \Phi_{j-1,k+1} + \Phi_{j,k+2} + \Phi_{j,k-2}\right) \qquad [2]$$

In the term of Λ, equation (1) can be written as following,

$$rI_{j,k} = 6V_{j,k} - 6\Lambda V_{j,k} \qquad [3]$$

Or

$$V_{j,k} = \Lambda V_{j,k} + \frac{rI_{j,k}}{6} \qquad [4]$$

The next step is to solve this equation by assuming that $V_{j,k}$ tends to zero at infinity. The resistance between point *(0,0)* and *(m,n)* can be calculated by the following formula,

$$R_{mn} = \frac{V_{00} - V_{mn}}{I} \qquad [5]$$

It has been proved by Giaro [1] that the following equation has a unique solution (at most one solution) and the solution tends to zero at infinity (for $0 < a \le 1$).

$$V_{j,k} = a\Lambda V_{j,k} + \frac{rI_{j,k}}{6} \qquad [6]$$

Furthermore, we are going to solve this equation, then for a fixed a, we will calculate the value of V_{00}-V_{jk} and take the limit for $a \to 1^-$ to obtain the desired equivalent resistance formula.

Let assume that there are exists a function of two real variables

$$F(x,y):[-p,p]x[-p,p] \to \Re$$

Expanding this function in a double Fourier series with the Fourier coefficient equal to the value of $V_{j,k}$ gives,

$$F(x,y) = \sum_{j,k} V_{j,k} e^{i(jx+ky)} \qquad [7]$$

Where,

$$V_{j,k} = \frac{1}{4p^2} \int_{-p}^{p}\int_{-p}^{p} F(x,y) e^{-i(jx+ky)} dxdy \qquad [8]$$

Using equation (6) and (7) we have,

$$F(x,y) = \sum_{j,k} \left(a\Lambda V_{j,k} + \frac{rI_{j,k}}{6} \right) e^{i(jx+ky)} \qquad [9]$$

After operating Λ, we will get,

$$F(x,y) = \sum_{j,k} \left(\frac{a}{6} \left(V_{j+1,k+1} + V_{j+1,k-1} + V_{j-1,k+1} + V_{j-1,k-1} + V_{j,k+2} + V_{j,k-2} \right) \right) e^{i(jx+ky)} \qquad [10]$$

Since $I_{j,k}$ has a value ($=I$) only at *(0,0)* and *(m,n)* then,

$$\sum_{j,k} \left(\frac{rI_{j,k}}{6} \right) e^{i(jx+ky)} = \frac{rI}{6} \left(1 - e^{i(mx+ny)} \right) \qquad [11]$$

Using equation (11) and some properties of summation $\sum$ (change indices), we have:

$$F(x,y)=\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)+\frac{a}{6}\sum_{j,k}V_{j,k}\left(e^{i((j-1)x+(k-1)y)}+e^{i((j-1)x+(k+1)y)}+e^{i((j+1)x+(k-1)y)}+e^{i((j+1)x+(k+1)y)}\right.$$
$$\left.+e^{i(jx,(k-2)y)}+e^{i(jx+(k+2)y)}\right)$$
$$=\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)+\frac{a}{6}\sum_{j,k}V_{j,k}e^{i(jx+ky)}\left(e^{i(-x-y)}+e^{i(-x+y)}+e^{i(x-y)}+e^{i(x+y)}+e^{i(-2y)}+e^{i(2y)}\right)$$

[12]

Simplifying the above equation by a trigonometry identity $e^{iq}+e^{-iq}=2\cos q$ gives,

$$F(x,y)=\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)+\frac{a}{6}\sum_{j,k}V_{j,k}e^{i(jx+ky)}\left(e^{-ix}2\cos y+e^{+ix}2\cos y+2\cos 2y\right)$$
$$=\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)+\frac{a}{6}\sum_{j,k}V_{j,k}e^{i(jx+ky)}\left(4\cos x\cos y+2\cos 2y\right) \qquad [13]$$
$$=\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)+\frac{a}{6}\left(4\cos x\cos y+2\cos 2y\right)F(x,y)$$

Solve this equation for *F(x,y)* to get,

$$F(x,y)=\frac{\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)}{1-\frac{a}{6}\left(4\cos x\cos y+2\cos 2y\right)} \qquad [14]$$

It is easy to notice that the function F is continuous, differentiable, and bounded. Consequently, it can be expressed in terms of Fourier component $V_{j,k}$.

From equation (8), the coefficient of the double Fourier series V_{00} and V_{mn} can be written as the following,

$$V_{00}=\frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}F(x,y)dxdy=\frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)}{1-\frac{a}{6}\left(4\cos x\cos y+2\cos 2y\right)}dxdy \qquad [15]$$

And

$$V_{mn} = \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p} F(x,y)e^{-i(mx+ny)}dxdy = \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\frac{rI}{6}\left(1-e^{i(mx+ny)}\right)e^{-i(mx+ny)}}{1-\frac{a}{6}(4\cos x\cos y+2\cos 2y)}dxdy$$

$$= \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\frac{rI}{6}\left(e^{-i(mx+ny)}-1\right)}{1-\frac{a}{6}(4\cos x\cos y+2\cos 2y)}dxdy \qquad [16]$$

Therefore,

$$V_{00}-V_{mn} = \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\frac{rI}{6}\left(2-e^{-i(mx+ny)}-e^{i(mx+ny)}\right)}{1-\frac{a}{6}(4\cos x\cos y+2\cos 2y)}dxdy$$

$$= \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\frac{rI}{6}\left(2-2\cos(mx+ny)\right)}{1-\frac{a}{6}(4\cos x\cos y+2\cos 2y)}dxdy \qquad [17]$$

If $a \to 1^-$ then the above formula gives the difference of V_{jk} satisfying the equation (4) at the points *(0,0)* and *(m,n)*. Then the resistance between those points can be evaluated by using equation (5). The results is,

$$R_{mn} = \frac{r}{12p}\int_{-p}^{p}\int_{-p}^{p}\frac{\left(1-\cos(mx+ny)\right)}{1-\frac{1}{6}(4\cos x\cos y+2\cos 2y)}dxdy \qquad [18]$$

The results for several *(m,n)* are given in table 1.

Table 1. The equivalent resistance between points with coordinates *(m,n)* of triangle infinite networks of resistor.

No	Coordinates(*m,n*)	Resistance/R(*m,n*)	No	Coordinates(*m,n*)	Resistance/R(*m,n*)
1	[-3,-7]	0,604	21	[-1,5]	0.513
2	[-3,-5]	0.57	22	[0,0]	0
3	[-3,-3]	0.536	23	[0,2]	0.333
4	[-3,-1]	0.513	24	[0,4]	0.461
5	[-3,1]	0.513	25	[0,6]	0.536
6	[-3,3]	0.536	26	[1,-7]	0.57
7	[-3,5]	0.57	27	[1,-5]	0.513
8	[-2,-6]	0.536	28	[1,-3]	0.436
9	[-2,-4]	0.513	29	[1,-1]	0.333
10	[-2,-2]	0.461	30	[1,1]	0.333
11	[-2,0]	0.436	31	[1,3]	0.436
12	[-2,2]	0.461	32	[1,5]	0.513
13	[-2,4]	0.513	33	[2,-6]	0.563
14	[-2,6]	0.563	34	[2,-4]	0.513
15	[-1,-7]	0.57	35	[2,-2]	0.461
16	[-1,-5]	0.513	36	[2,0]	0.436
17	[-1,-3]	0.436	37	[2,2]	0.461
18	[-1,-1]	0.333	38	[2,4]	0.513
19	[-1,1]	0.333	39	[2,6]	0.563
20	[-1,3]	0.436	40	[3,-7]	0.605

Discussion

Atkinson and Steenwijk [2] used separation variable method and Cserti [3] utilized Green function method, found that the formula for resistance in the triangle case is the following,

$$R_{m'n'} = r\int_0^{\frac{p}{2}} \frac{dx}{p} \frac{1-e^{-|n'-m'|x}\cos(n'+m')}{\sinh x \cos y} \quad [19]$$

They used a hexagonal coordinate system and the definition of *(m',n')* is shown in the Figure 2.

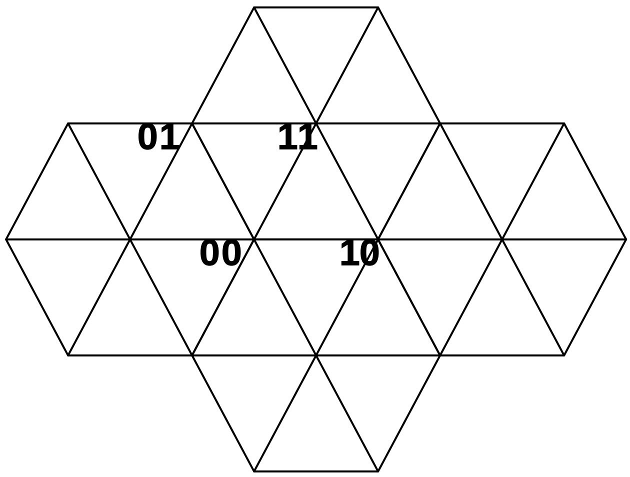

Figure 2. Coordinates in the hexagonal / triangle coordinate system [2,3]

To see whether equation (18) and equation (19), we calculate numerically for several points (we have to make sure the compared points are at the same points). Table 2 shows that both formulas give exactly the same results.

Table 2. Comparison results from equation (18) and equation (19)

No	Resistance(R_{mn}/r)	
	Equation (18) Coordinate System	***Equation (19) Coordinates System***
1	[1,1] = 0.333	[0,1] = $\frac{1}{3}$
2	[0,2] = 0.436	[1,2] = $-\frac{2}{3}+\frac{2\sqrt{3}}{p}$
3	[3,1] = 0.513	[1,3] = $-5+\frac{10\sqrt{3}}{p}$
4	[2,2] = 0.461	[0,2] = $\frac{8}{3}-\frac{4\sqrt{3}}{p}$
5	[3,3] = 0.536	[0,3] = $27-\frac{48\sqrt{3}}{p}$

3. Infinite hexagonal lattice networks of identical resistors (honeycomb lattice)

In this section, I the resistance between any two points in the infinite hexagonal lattices networks of identical resistor. Atkinson and Steenwijk [4] have showed it and that the hexagonal lattice could to be constructing from the triangular lattice by the application of the so-called ΔY transformation showed it has.

We are going to calculate the resistance using two approaches: 1) Using ΔY transformation and the result from triangle case; 2) Using a direct calculation from the hexagonal lattices. We will show that both calculations give the same results.

3.2.1 Method 1: ΔY transformation

A triangle made out of three $\boldsymbol{r}$ ohm resistors, can be transformed into **Y** form made out of three $\boldsymbol{r}/3$ resistor, in the sense that the external currents, $\boldsymbol{I}_1$, $\boldsymbol{I}_2$, and $\boldsymbol{I}_3$, and the peripheral potential $\boldsymbol{V}_1$,`$\boldsymbol{V}_2$ and $\boldsymbol{V}_3$ are the same. In the **Y** form of potential in the mid point is $V_0 = (V_1+V_2+V_3)/3$ on condition that no current enters or leaves the circuit at this point from the outside.

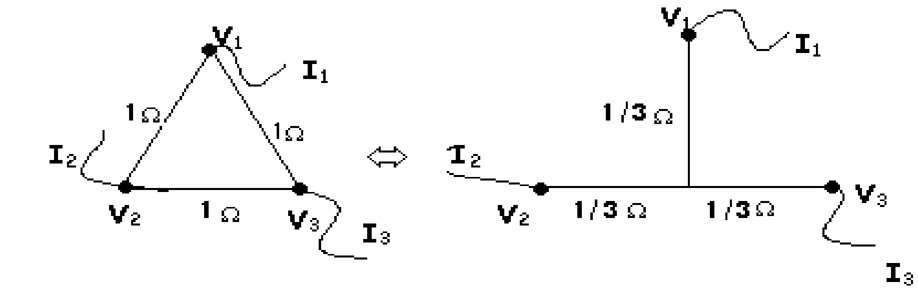

Figure 3 ΔY transformation

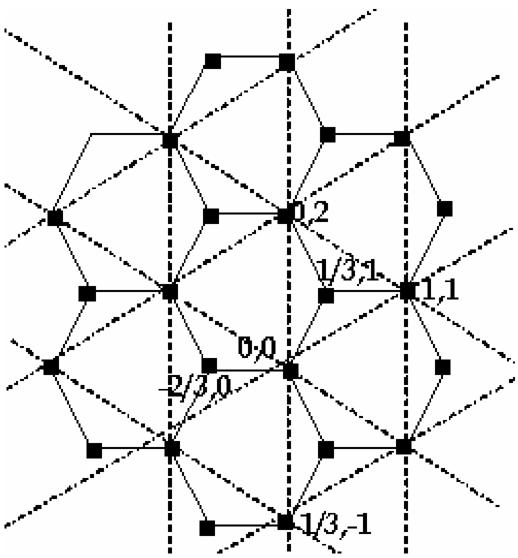

Figure 4 infinite hexagonal lattice

Figure 4 show how to produce a hexagonal lattice by using ΔY transformation repeatedly applied to triangular lattices. Note that the equivalent resistors in this case are all ***r*/3**, so we will need to multiply the final answer for R_{mn} by 3 in order to renormalize to the desired case of a hexagonal lattice of ***r*** ohm resistor. In figure 4 , I retain the same coordinates for the points that are common to the triangular lattice and have assigned the relevant fractional m and n values to the new lattice points, the middle of Y's.

Assume that a current I entering the node *(0,0)* and leaving the node *(m,n)* as we have in triangular case. The potentials at the node of hexagonal (not at the node of triangle are given by

$$V_{m+1/3,n+1} = \frac{1}{3}\left[V_{mn} + V_{m,n+2} + V_{m+1,n+1}\right] \qquad [20]$$

The potential difference between the origin and the node *(m,n)* is V_{mn}-V_{00} but since we applies to a hexagonal lattice of 1/3 ohm resistors, the final results for a lattice of 1 ohm resistors is 3 times of this value,

$$\begin{aligned} R_{m+1/3,n+1} &= 3.\frac{1}{3}\left[\frac{V_{00}-V_{m+1/3,n+1}}{I}\right] \\ &= \frac{1}{3}\left[\frac{(V_{00}-V_{mn})+(V_{00}-V_{mn+2})+(V_{00}-V_{m+1,n+1})}{I}\right] \\ &= \frac{1}{3}\left[R_{mn}+R_{m,n+2}+R_{m+1,n+1}\right] \end{aligned} \qquad [21]$$

Table 3 shows the resistances calculated from equation (21) and the results quoted from Atkinson and Steenwijk [2].

Table 3 Resistance between points *(0,0)* and point *(m,n)* in hexagonal lattice networks of identical resistors

No	From Equation (21)	Reference [2]
1	$R_{0,1} = R_{0,-1} = R_{1,0} = 0{,}667\ r$	$2r/3$
2	$R_{1,-1} = R_{1,1} = R_{-1,1} = R_{-1,-1} = r$	r
3	$R_{-1,0} = R_{1,2} = 1{,}103\ r$	$\frac{2\sqrt{3}}{p}r$
4	$R_{-1,-2} = R_{-1,2} = R_{2,-1} = R_{2,1} = 1{,}231\ r$	$\left(\frac{7}{3}-\frac{2\sqrt{3}}{p}\right)r$
5	$R_{-1,-3} = R_{-1,3} = R_{-1,-3} = R_{-1,3} = 1{,}308\ r$	$\left(-2+\frac{6\sqrt{3}}{p}\right)r$
6	$R_{-2,2} = R_{-2,-2} = R_{2,2} = R_{2,-2} = 1{,}385\ r$	$\left(8-\frac{12\sqrt{3}}{p}\right)r$
7	$R_{2,-3} = R_{2,3} = R_{-1,1} = R_{-1,-1} = 1{,}411\ r$	$\left(-3+\frac{8\sqrt{3}}{p}\right)r$
8	$R_{-1,-4} = R_{-1,4} = 1{,}463\ r$	$\left(\frac{-32}{3}+\frac{22\sqrt{3}}{p}\right)r$

3.2.1 Method 2: Double Fourier series method

Consider a current of I amperes enter the *(0,0)* node from a source outside the hexagonal lattices and leaves at *(m,n)*. To simplify let I draw the infinite hexagonal network in the form as shown in Fig. 5.

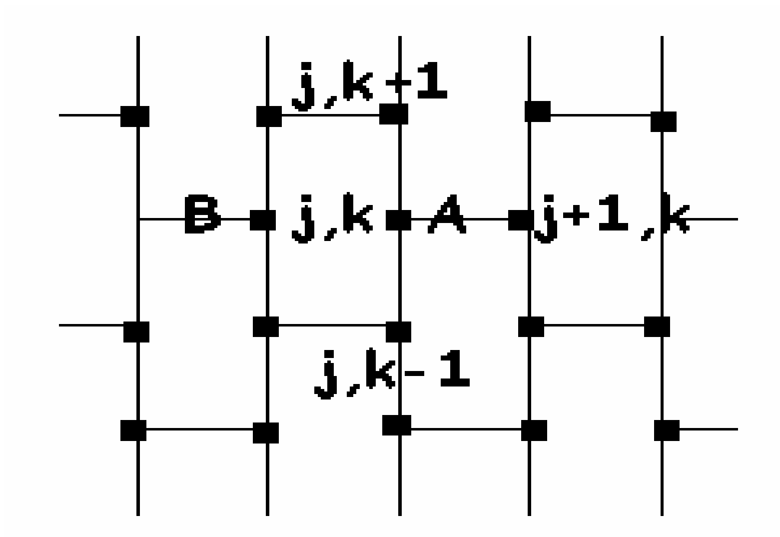

Figure 5. Infinite “modified” hexagonal network

Using Kirchoff’s first rule and Ohm’s law at point A, we have,

$$\begin{aligned} rI_{j,k} &= \left(V_{jk} - V_{j,k+1}\right) + \left(V_{jk} - V_{j,k-1}\right) + \left(V_{jk} - V_{j+1,k}\right) \\ &= 3V_{jk-}\left(V_{j,k+1} + V_{j,k-1} + V_{j+1,k}\right) \end{aligned} \quad [22]$$

Or,

$$\left(V_{jk} = \left(V_{j,k+1} + V_{j,k-1} + V_{j+1,k}\right) + \frac{rI_{j,k}}{3} \right)_{A(j,k)} \quad [23]$$

However, equation (23) will slightly change f we take B as point *(j,k)*

$$\left(V_{jk} = \left(V_{j,k+1} + V_{j,k-1} + V_{j-1,k}\right) + \frac{rI_{j,k}}{3}\right)_{B(j,k)} \qquad [24]$$

From equation (23) and (24) we conclude that the potential will depend on whether *(j+k)* is even or odd. We will use equation (23) for *(j+k)* is even and equation (24) for *(j+k)* is odd.

Now, let I introduce two double Fourier series,

$$F_1(x,y) = \sum_{j+k=even} V_{jk} e^{i(jx+ky)} \qquad [25]$$

And

$$F_1(x,y) = \sum_{j+k=odd} V_{jk} e^{i(jx+ky)} \qquad [26]$$

The F_1*(x,y)* is for *(j+k)* even and F_2*(x,y)* is for *(j+k)* odd.

Since the potentials depend on the oddness or evenness *(j+k)* then we have to split our analysis: a) Case 1 when *(m+n)* even and b) Case 2 when *(m+n)* odd

3.2.1 Case 1: *(m+n)* is even

$$F_1(x,y) = \sum_{(j+k=even)} \left(\frac{1}{3}\left(V_{j,k+1} + V_{j,k-1} + V_{j+1,k}\right) + \frac{rI_{j,k}}{3}\right) e^{i(jx+ky)} \qquad [27]$$

Since $I_{j,k}$ equal to *I* at *(0,0)* and *–I* at *(m,n)* then,

$$\sum_{j+k=even} \left(\frac{rI_{j,k}}{3}\right) e^{i(jx+ky)} = \frac{rI}{3}\left(1 - e^{i(mx+ny)}\right) \qquad [28]$$

Using equation (27) and the properties of $\sum$ (change the indices), yields,

$$\begin{aligned} F_1(x,y) &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\sum_{j+k=odd} V_{jk}\left(e^{i(jx+(k-1)y)}+e^{i(jx+(k+1)y)}+e^{i((j-1)x+ky)}\right) \\ &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\sum_{j+k=odd} V_{jk}e^{i(jx+ky)}\left(e^{i(-y)}+e^{i(y)}+e^{-i(x)}\right) \\ &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\left(2\cos y+e^{-ix}\right)\sum_{j+k=odd} V_{jk}e^{i(jx+ky)} \\ &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\left(2\cos y+e^{-ix}\right)F_2(x,y) \end{aligned} \qquad [29]$$

Now let's work for $F_2(x,y)$

$$F_2(x,y)=\sum_{(j+k=odd)}\left(\frac{1}{3}\left(V_{j,k+1}+V_{j,k-1}+V_{j-1,k}\right)+\frac{rI_{j,k}}{3}\right)e^{i(jx+ky)} \qquad [30]$$

Since the current is 0 for $(j+k)$ odd, and by changing the indices, we get,

$$\begin{aligned} F_2(x,y) &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\sum_{j+k=odd} V_{jk}\left(e^{i(jx+(k-1)y)}+e^{i(jx+(k+1)y)}+e^{i((j+1)x+ky)}\right) \\ &= \frac{1}{3}\sum_{j+k=even} V_{jk}e^{i(jx+ky)}\left(e^{i(-y)}+e^{i(y)}+e^{i(x)}\right) \\ &= \frac{1}{3}\left(2\cos y+e^{ix}\right)\sum_{j+k=even} V_{jk}e^{i(jx+ky)} \\ &= +\frac{1}{3}\left(2\cos y+e^{-ix}\right)F_1(x,y) \end{aligned} \qquad [31]$$

Replacing $F_2(x,y)$ in equation (29) by equation (31) will give,

$$\begin{aligned} F_1(x,y) &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{3}\left(2\cos y+e^{-ix}\right)\frac{1}{3}\left(2\cos y+e^{ix}\right)F_1(x,y) \\ &= \frac{rI}{3}\left(1-e^{i(mx+ny)}\right)+\frac{1}{9}\left(4\cos^2 y+1+4\cos y\cos x\right)F_1(x,y) \end{aligned} \qquad [32]$$

Solve this equation to get $F_1(x,y)$

$$F_1(x,y) = \frac{\frac{rI}{3}\left(1 - e^{i(mx+ny)}\right)}{1 - \frac{1}{9}\left(4\cos^2 y + 4\cos y \cos x + 1\right)} = \frac{3rI\left(1 - e^{i(mx+ny)}\right)}{8 - 4\cos y(\cos y + \cos x)} \qquad [33]$$

The resistance $\boldsymbol{R_{mn}}$ can be calculated as following,

$$R_{mn} = \frac{V_{00} - V_{mn}}{I} = \frac{\frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p} F_1(x,y)dxdy - \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p} F_1(x,y)e^{-i(mx+ny)}dxdy}{I} \qquad [34]$$

Or,

$$R_{mn} = \frac{3r}{4p^2}\int_{-p}^{p}\int_{-p}^{p} \frac{\left(1 - e^{i(mx+ny)}\right)\left(1 - e^{-i(mx+ny)}\right)}{8 - 4\cos y(\cos y + \cos x)} dxdy \qquad [35]$$

After a little algebra and a small trigonometry manipulation, we will get

$$R_{mn} = \frac{3r}{4p^2}\int_{-p}^{p}\int_{-p}^{p} \frac{\left(1 - \cos(mx + ny)\right)}{4 - 2\cos y(\cos y + \cos x)} dxdy \text{ (for } (m+n) \text{ even)} \qquad [36]$$

3.2.2. Case: *(m+n)* is odd

Here we have to be careful in treating $F_1(x,y)$ and $F_2(x,y)$. The current term in equation (27) will give $\frac{rI}{3}$ and equation (30) will give $\frac{-rI}{3}e^{i(mx+ny)}$. Therefore, $F_1(x,y)$ and $F_2(x,y)$ can be written as:

$$F_1(x,y) = \frac{rI}{3} + \frac{1}{3}\left(2\cos y + e^{-ix}\right)F_2(x,y) \qquad [37]$$

And

$$F_2(x,y) = -\frac{rI}{3}e^{i(mx+ny)} + \frac{1}{3}\left(2\cos y + e^{ix}\right)F_1(x,y) \qquad [38]$$

Solving these two equations gives

$$F_1(x,y) = \frac{rI\left(3 - e^{i(mx+ny)}\left(2\cos y + e^{ix}\right)\right)}{8 - 4\cos y(\cos y + \cos x)} \qquad [39]$$

And

$$F_2(x,y) = \frac{rI\left(-3e^{i(mx+ny)} + \left(2\cos y + e^{-ix}\right)\right)}{8 - 4\cos y(\cos y + \cos x)} \qquad [40]$$

Given these two Fourier function, we can determine voltage at any point that is:
For *(j+k)* is even

$$V_{jk} = \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{rI\left(3 - e^{i(mx+ny)}\left(2\cos y + e^{-ix}\right)\right)}{8 - 4\cos y(\cos y + \cos x)}e^{-i(jx+ky)}dxdy \qquad [41]$$

And for *(j+k)* odd

$$V_{jk} = \frac{1}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{rI\left(-3e^{i(mx+ny)} + \left(2\cos y + e^{-ix}\right)\right)}{8 - 4\cos y(\cos y + \cos x)}e^{-i(jx+ky)}dxdy \qquad [42]$$

The resistance R_{mn} can be calculated from $R_{mn} = \frac{V_{00} - V_{mn}}{I}$.
Since *0+0* is even we use equation (41) to calculate V_{00} and equation (42) to calculate V_{mn} (remember that we are working for case *m+n* is odd)

After some algebra and trigonometry tricks, finally we could find $\boldsymbol{R_{mn}}$ as follows,

$$R_{mn} = \frac{r}{4p^2}\int_{-p}^{p}\int_{-p}^{p}\frac{\left(3 - 2\cos y\cos(mx+ny) - \cos((m-1)x + ny)\right)}{4 - \cos y(\cos y + \cos x)}dxdy \text{ (For (m+n) odd)}$$

[43]

Numerical result

It would be very interesting to investigate if I could calculate the integral analytically. I tried Maple 9 to carry out the calculation, however it will take very long time to get the result especially when the points are separated far away $(m,n>>1)$.

I also try to calculate the integration numerically. I started by using Simpson's method with n equal spacing for each dimensions of integration. For two-dimensional integration, the number of interval to calculate is n^2. Therefore, the computational time will increase even faster as I increase the number of intervals n order to improve the accuracy. This problem become especially significant if I want to calculate equivalent resistance between points that separate far away $(m,n>>1)$. The other problem is cos $(mx+ny)$ oscillates rapidly and thus requires more sampling point to give reasonable accurate result. To solve this problem, I implement Gaussian quadrature integration. The fundamental theorem of Gaussian quadrature states that the optimal abscissas of the m-point Gaussian quadrature formulas are precisely the roots of orthogonal polynomial for the same interval and weighting function. Gaussian quadrature is optimal because it fits all polynomials up to degree $2m$ exactly. This type of numerical of integration is more efficient since it requires much fewer sampling point to obtain the same accuracy. The numerical routine is implementation in MATLAB, which is powerful to do many intensive calculations. In carrying out the integration, one has to be pay attention to the singular point in the integrand. I could avoid this singular point by choosing the right sampling point in the integration. This is another advantage of Gaussian quadrature.

The numerical value of the equivalent resistance between points with coordinates (m,n) are given in table 4 for $(m+n)$ is odd and in table 5 for $(m+n)$ is even.

Table 4. The resistance for *(m+n)* is odd (multiply by ***r***)

[-5,-4]=1.89727	[-2,1]=1.4118	[2,-5]=1.59089
[-5,-2]=1.86245	[-2,3]=1.51238	[2,-3]=1.41187
[-5,-0]=1.84972	[-2,5]=1.64799	[2,-1]=1.2316
[-5,2]=1.86243	[-1,-4]=1.46381	[2,1]=1.2316
[-5,4]=1.89725	[-1,-2]=1.23163	[2,3]=1.41187
[-4,-5]=1.83619	[-1,0]=1.1034	[2,5]=1.5909
[-4,-3]=1.77581	[-1,2]=1.23163	[3,-4]=1.62141
[-4,-1]=1.73995	[-1,4]=1.46381	[3,-2]=1.51238
[-4,1]=1.73995	[0,-5]=1.51181	[3,0]=1.46389
[-4,3]=1.7758	[0,-3]=1.23134	[3,2]=1.51239
[-4,5]=1.83616	[0,-1]=0.66702	[3,4]=1.62142
[-3,-4]=1.69758	[0,1]=0.667016	[4,-5]=1.7758
[-3,-2]=1.62143	[0,3]=1.23134	[4,-3]=1.69756
[-3,0]=1.59098	[0,5]=1.5118	[4,-1]=1.64797
[-3,2]=1.62142	[1,-4]=1.41158	[4,1]=1.64797
[-3,4]= 1.69756	[1,-2]=1.10332	[4,3]=1.69757
[-2,-5]=1.648	[1,0]=0.667122	[4,5]=1.77582
[-2,-3]=1.51238	[1,2]=1.10333	
[-2,-1]=1.41185	[1,4]=1.41158	

Table 5. The resistance for *(m+n)* is even (multiply by ***r***)

[-5,-5]=1.89352	[-2,0]=1.30884	[1,3]=1.30893
[-5,-3]=1.84518	[-2,2]=1.38528	[1,5]=1.54111
[-5,-1]=1.81737	[-2,4]=1.54139	[2,-4]=1.54139
[-5,1]=1.81737	[-1,-5]=1.54111	[2,-2]=1.38528
[-5,3]=1.84516	[-1,-3]=1.30893	[2,0]=1.30884
[-5,5]=1.89349	[-1,-1]=1.00068	[2,2]=1.38529
[-4,-4]=1.76986	[-1,1]=1.00068	[2,4]=1.5414
[-4,-2]=1.7214	[-1,3]=1.30892	[3,-5]=1.71239
[-4,0]=1.6901	[-1,5]=1.5411	[3,-3]=1.61045
[-4,2]=1.71239	[0,-4]=1.38471	[3,-1]=1.54141
[-4,4]=1.76984	[0,-2]=1.00037	[3,1]=1.54142
[-3,-5]=1.71241	[0,0]=0	[3,3]=1.61046
[-3,-3]=1.61046	[0,2]=1.00037	[3,5]=1.71241
[-3,-1]=1.54142	[0,4]=1.38471	[4,-4]=1.76984
[-3,1]=1.54141	[1,-5]=1.5411	[4,-2]=1.71239
[-3,3]= 1.61045	[1,-3]=1.30892	[4,0]=1.6901
[-3,5]=1.71239	[1,-1]=1.00068	[4,2]=1.7124
[-2,-2]=1.38529	[1,1]=1.00068	[4,4]=1.76986

Discussion

It is interesting to compare the results above with the results that we get from the ΔY transformation or the Atkinson and Steewijk's [2]. We can check easily that from table 3 and 4 or 5 will be more accurate if we use more Gauss points in the numerical calculation.

Acknowledgement

I would like to express my gratitude to Prof. Yohanes Surya Ph.D for proposing me the subjects of this project and valuable discussion and to I Made Agus Wirawan for helping me in learning numerical calculation. To my brother Yoppi, Frengki,Agus,Apilen and to my Father and Mother, thank you very much.

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